W6. (Solution by the proposer.) From

$$r = 4R \cdot \sin\frac{A}{2} \cdot \sin\frac{B}{2} \cdot \sin\frac{C}{2} \Rightarrow \frac{r}{R} =$$

$$= 4 \cdot \sqrt{\frac{(p-b)(p-c)(p-c)(p-a)(p-a)(p-b)}{a^2b^2c^2}} =$$

$$= 4 \cdot \frac{(p-a)(p-b)(p-c)}{abc} = 4 \cdot \frac{(b+c-a)(c+a-b)(b+c-a)}{8abc} \Rightarrow$$

$$\Rightarrow \frac{r}{R} = \frac{(b+c-a)(c+a-b)(b+c-a)}{2abc}$$

With the help of Ravi substitutions, $a = y+z$, $b = x+z$, $c = x+y$, demonstrated inequality becomes:

$$\frac{y+z}{2x+y+z} + \frac{x+z}{2y+x+z} + \frac{x+y}{2z+x+y} + \frac{8xyz}{2(x+y)(y+z)(z+x)} \leq 2 \Leftrightarrow$$

$$\Leftrightarrow \frac{y+z}{2x+y+z} + \frac{x+z}{2y+x+z} + \frac{x+y}{2z+x+y} + \frac{4xyz}{(x+y)(y+z)(z+x)} \leq 2$$

Then,

$$\frac{x^2}{(x+y)(x+z)} + \frac{y^2}{(y+z)(y+x)} \geq \frac{(x+y)^2}{(x+y)(x+y+2z)} = \frac{x+y}{x+y+2z}$$

Analogous,

$$\frac{y^2}{(y+z)(y+x)} + \frac{z^2}{(z+x)(z+y)} \geq \frac{y+z}{2x+y+z}$$

and

$$\frac{z^2}{(z+x)(z+y)} + \frac{x^2}{(x+y)(x+z)} \geq \frac{z+x}{2y+x+z}$$

Summing up,

$$2\left(\frac{x^2}{(x+y)(x+z)}+\frac{y^2}{(y+z)(y+x)}+\frac{z^2}{(z+x)(z+y)}\right)\geq$$

$$\geq\frac{x+y}{x+y+2z}+\frac{y+z}{2x+y+z}+\frac{z+x}{2y+x+z}\Rightarrow$$

$$\Rightarrow\frac{2\left(x^2z+x^2y+y^2z+z^2x+z^2y\right)}{(x+y)(y+z)(z+x)}\geq\frac{x+y}{x+y+2z}+\frac{y+z}{2x+y+z}+\frac{z+x}{2y+x+z}\Leftrightarrow$$

$$\Leftrightarrow\frac{2\left[(x+y)(y+z)(z+x)-2xyz\right]}{(x+y)(y+z)(z+z)}\geq\frac{x+y}{x+y+2z}+\frac{y+z}{2x+y+z}+\frac{z+x}{2y+x+z}\Rightarrow$$

$$\Rightarrow 2-\frac{4xyz}{(x+y)(y+z)(z+z)}\geq\frac{x+y}{x+y+2z}+\frac{y+z}{2x+y+z}+\frac{z+x}{2y+x+z}\Rightarrow$$

$$\Rightarrow\frac{y+z}{2x+y+z}+\frac{x+z}{2y+x+z}+\frac{x+y}{2z+x+y}+\frac{4xyz}{(x+y)(y+z)(z+x)}\leq 2,$$

which ends demonstration.

Second solution. Let $x:=s-a,y:=s-b,z:=s-c$ where s is semiperimeter of the triangle. Then $x,y,z>0$. Assume (due homogeneity of original inequality) $s=1$ and let $p:=xy+yz+zx$, $q:=xyz$. Then $x+y+z=1,a=1-x,b=1-y,c=1-z$, $abc=p-q,r=sr=[ABC]=\sqrt{q}$,

$$R=\frac{abc}{4[ABC]}=\frac{p-q}{4\sqrt{q}},\sum_{cyc}\frac{a}{b+c}=\sum_{cyc}\frac{1-x}{1+x}=\sum_{cyc}\frac{2-(1+x)}{1+x}=2\sum_{cyc}\frac{1}{1+x}-3=$$

$$=\frac{2\sum_{cyc}(1+y)(1+z)}{(1+x)(1+y)(1+z)}-3=\frac{2\sum_{cyc}(2-x+yz)}{(1+x)(1+y)(1+z)}-3=\frac{2(5+p)}{2+p+q}-3$$

and original inequality becomes

$$\frac{2(5+p)}{2+p+q}-3+\frac{4q}{p-q}\leq 2\iff\frac{2(5+p)}{2+p+q}+\frac{4q}{p-q}\leq 5\iff$$

$$\Longleftrightarrow 2(5+p)(p-q)+4q(2+p+q)\leq 5(2+p+q)(p-q) \Longleftrightarrow$$

$$\Longleftrightarrow 9q^2+2q(p+4)-3p^2\leq 0$$

Since

$$3p=3(xy+yz+zx)\leq (x+y+z)^2=1$$

and

$$p^2=(xy+yz+zx)^2\geq 3xyz(x+y+z)=3q$$

then

$$9q^2+2q(p+4)-3p^2\leq 9\left(\frac{p^2}{3}\right)^2+2\cdot\frac{p^2}{3}(p+4)-3p^2=-\frac{p^2(p+1)(1-3p)}{3}\leq 0.$$

Arkady Alt

Third solution. After calculations we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{r}{R}\leq 2\Leftrightarrow$$

$$\Leftrightarrow \sum aR(a+b)(c+a)+r\prod(a+b)\leq 2R\prod(a+b)\Leftrightarrow$$

$$\Leftrightarrow R\sum a^3+(r-R)\sum a\sum ab+(2R-r)abc\leq 0$$

With usual notations, we get

$$a+b+c=2s, abc=aRrs, \sum a^3=2s\left(s^2-3r^2-6Rr\right), \sum ab=a^2+r^2+4Rr$$

$$R\sum a^3+(r-R)\sum a\sum ab+(2R-r)abc\leq 0\Leftrightarrow$$

$$\Leftrightarrow 2sR\left(s^2-3r^2-6Rr\right)+2s(r-R)\left(s^2+r^2+4Rr\right)+4Rrs(2R-r)\leq 0\Leftrightarrow$$

$$\Leftrightarrow 2sr\left(s^2+r^2-2Rr-6R^2\right)\leq 0$$

Applying Gerretsen's inequality, we obtain

$$s^2 \leq 4R^2 + 4Rr + 3r^2$$

$$R^2 - Rr - 2r^2 \geq 0 \Rightarrow (R - 2r)(R + r) \geq 0,$$

that is true.

Nicuşor Zlota

W7. (Solution by the proposer.) Note $\beta\gamma = a$, $\alpha\gamma = b$, $\alpha\beta = c$. For the first part of inequality:

$$\sin\alpha + \sin\beta + \sin\gamma - \sin\alpha \cdot \sin\beta \cdot \sin\gamma \geq \sin^3\alpha + \sin^3\beta + \sin^3\gamma \Leftrightarrow$$

$$\Leftrightarrow \frac{a+b+c}{2R} - \frac{abc}{8R^3} \geq \frac{a^3+b^3+c^3}{8R^3} \Leftrightarrow 4R^2(a+b+c) - abc \geq a^3 + b^3 + c^3 \Leftrightarrow$$

$$\underset{abc=4Rrp}{\overset{\sum a^3 = 2p(p^2-3r^2-6rR)}{\Leftrightarrow}} 4R^2 \cdot 2p - 4Rrp \geq 2p\left(p^2 - 3r^2 - 6Rr\right) \Leftrightarrow$$

$$\Leftrightarrow 8R^2 - 4Rr \geq 2p^2 - 6r^2 - 12rR \Leftrightarrow 2p^2 \leq 8rR + 8R^2 - 4Rr \Leftrightarrow$$

$$\Leftrightarrow p^2 \leq 4R^2 + 4rR + 3r^2,$$

which represents inequality Gerretsen.
a). Suppose $\max(\alpha, \beta, \gamma) < \frac{\pi}{2}$. For starters we demonstrate the following lemma:
Lemma. If M is a point in plan of triangle ABC, then the inequality

$$AB \cdot BC \cdot CA \leq MA^2 \cdot BC + MB^2 \cdot AC + MC^2 \cdot BC$$

Proof. In the complex plane either x, y, z affixes points A, B, C, and M the landmark XOY. Using equality

$$x^2(y-z) + y^2(z-x) + z^2(x-y) = (x-y)(y-z)(z-x)$$

we can write

$$|(x-y)(y-z)(z-x)| = \left|x^2(y-z) + y^2(z-x) + z^2(x-y)\right| \leq$$

$$\leq |x|^2 |y-z| + |y|^2 |z-x| + |z|^2 |x-y| ,$$

from where

$$AB \cdot BC \cdot CA \leq MA^2 \cdot BC + MB^2 \cdot AC + MC^2 \cdot BC$$

Now, let H orthocentre of triangle $\alpha\beta\gamma$ ($H \in Int(\triangle ABC)$), and D, E, F the means sides $\beta\gamma, \alpha\gamma$ respectively $\alpha\beta$. According to the Lemma, used for the triangle DEF and $H \in (DEF)$, we have:

$$DE \cdot EF \cdot FD \leq HD^2 \cdot EF + HE^2 \cdot DF + HF^2 \cdot DE \tag{1}$$

Clear $EF = \frac{a}{2}, DE = \frac{c}{2}, DF = \frac{b}{2}$ in triangle $H\beta\gamma$, applying median theorem, we have

$$HD^2 = R^2 \left[2\left(\cos^2\beta + \cos^2\gamma\right) - \sin^2\alpha\right]$$

Considering that in a triangle we have identity

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 - 2\cos\alpha \cdot \cos\beta \cdot \cos\gamma,$$

obtain that

$$HD^2 = R^2 \left[\sin^2\alpha - 4\cos\alpha \cdot \cos\beta \cdot \cos\gamma\right]$$

Substituting in (1) and analogous, we get:

$$\frac{abc}{8} \leq R^2 \left[\sin^2\alpha - 4\cos\alpha \cdot \cos\beta \cdot \cos\gamma\right] \cdot \frac{a}{2} + R^2 \left[\sin^2\beta - 4\cos\alpha \cdot \cos\beta \cdot \cos\gamma\right] \cdot \frac{b}{2} +$$

$$+R^2 \left[\sin^2\gamma - 4\cos\alpha \cdot \cos\beta \cdot \cos\gamma\right] \cdot \frac{c}{2} \Rightarrow R^3 \sin\alpha \sin\beta \sin\gamma \leq$$

$$\leq R^3 \left(\sin^3\alpha + \sin^3\beta + \sin^3\gamma - 4\cos\alpha \cdot \cos\beta \cdot \cos\gamma \left(\sin\alpha + \sin\beta + \sin\gamma\right)\right),$$

from where

$$\sin^3\alpha + \sin^3\beta + \sin^3\gamma \geq \sin\alpha \sin\beta \sin\gamma + 4\cos\alpha \cdot \cos\beta \cdot \cos\gamma \left(\sin\alpha + \sin\beta + \sin\gamma\right)$$

b). If $\max(\alpha, \beta, \gamma) \geq \frac{\pi}{2}$, then $\cos\alpha \cdot \cos\beta \cdot \cos\gamma \leq 0$. Thus

$$\sin^3\alpha + \sin^3\beta + \sin^3\gamma \geq 3\sqrt[3]{\sin^3\alpha \cdot \sin^3\beta \cdot \sin^3\gamma} = 3\sin\alpha\sin\beta\sin\gamma >$$

$$> \sin\alpha\sin\beta\sin\gamma + 4\cos\alpha \cdot \cos\beta \cdot \cos\gamma \cdot (\sin\alpha + \sin\beta + \sin\gamma)$$

Second solution. Let a, b, c be sidelengths of the triangle and R, r, s be circumradius, inradius and semiperimeter, respectively. Then

$$\sum_{cyc} \sin A - \prod_{cyc} \sin A \geq$$

$$\geq \sum_{cyc} \sin^3 A \geq \prod_{cyc} \sin A + 4\prod_{cyc} \cos A \cdot \sum_{cyc} \sin A \iff 8R^3$$

$$8R^3 \sum_{cyc} \sin A - 8R^3 \prod_{cyc} \sin A \geq$$

$$\geq 8R^3 \sum_{cyc} \sin^3 A \geq 8R^3 \prod_{cyc} \sin A + 4\prod_{cyc} \cos A \cdot 8R^3 \sum_{cyc} \sin A \iff$$

$$8R^2 s - abc \geq a^3 + b^3 + c^3 \geq abc + 32R^2 s \prod_{cyc} \cos A \iff$$

$$8R^2 s - abc \geq a^3 + b^3 + c^3 \geq abc + 32R^2 s \cdot \frac{s^2 - (2R+r)^2}{4R^2} \iff$$

$$8R^2 s - abc \geq a^3 + b^3 + c^3 \geq abc + 8s\left(s^2 - (2R+r)^2\right) \iff$$

$$8R^2 s - 4Rrs \geq 2s\left(s^2 - 6Rr - 3r^2\right) \geq 4Rrs + 8s\left(s^2 - (2R+r)^2\right) \iff$$

$$4R^2 - 2Rr \geq s^2 - 6Rr - 3r^2 \geq 2Rr + 4\left(s^2 - (2R+r)^2\right) \qquad (1)$$

I. $4R^2 - 2Rr \geq s^2 - 6Rr - 3r^2 \iff s^2 \leq 4R^2 + 4Rr + 3r^2$ (Geretsen's Inequality).

II. $s^2 - 6Rr - 3r^2 \geq 2Rr + 4\left(s^2 - (2R+r)^2\right) \iff 3s^2 \leq 16R^2 + 8Rr + r^2$.

Since $s^2 \leq 4R^2 + 4Rr + 3r^2$ and $R \geq 2r$ (Euler's Inequality) then

$$16R^2 + 10Rr + r^2 - 3s^2 =$$

$$16R^2+8Rr+r^2-3\left(4R^2+4Rr+3r^2\right)+3\left(4R^2+4Rr+3r^2-s^2\right)=$$

$$4\left(R-2r\right)\left(R+r\right)+3\left(4R^2+4Rr+3r^2-s^2\right)\geq 0.$$

Arkady Alt

Third solution. With usual notations, we have
1).

$$\frac{a+b+c}{2R}-\frac{abc}{8R^3}\geq\frac{a^3+b^3+c^3}{8R^3}\Leftrightarrow 4R^22s-4Rrs\geq 2s\left(s^2-3r^2-6Rr\right)\Leftrightarrow$$

$$\Leftrightarrow 2s\left(4R^2-2Rr-s^2+3r^2+6Rr\right)\geq 0\Rightarrow s^2\leq 4R^2+3r^2+4Rr$$

which is Gerretsen's inequality
2).

$$\frac{a^3+b^3+c^3}{8R^3}\geq\frac{abc}{8R^3}+4\frac{s^2-(2R+r)^2}{4R^2}\cdot\frac{a+b+c}{2R}\Leftrightarrow 2s\left(-3s^2+16R^2+8Rr+r^2\right)\geq$$

Applying inequality Gerretsen, obtain

$$R^2-Rr-2r^2\geq 0\Rightarrow\left(R-2r\right)\left(R+r\right)\geq 0,$$

that is true.

Nicuşor Zlota

W8. (Solution by the proposer.) If $f(x)=0$, $\forall x\in[0,1]$, the conclusion is clear. Suppose there $x_0\in(0,1]$ such that

$$f\left(x_0\right)\neq 0\Rightarrow\int\limits_0^1 f\left(x\right)dx>0$$

We note $t=\frac{\int_0^1 g(x)dx}{\int_0^1 f(x)dx}dx\geq 0$. Then